

ZIEGLER'S MULTI-REFLECTION ARRANGEMENTS ARE FREE

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ABSTRACT. In 1989, Ziegler introduced the concept of a multi-arrangement. One natural example is the reflection arrangement of a unitary reflection group with multiplicity given by the number of reflections associated with each hyperplane. For all but three irreducible groups, Ziegler showed that each such multi-reflection arrangement is free. We complete Ziegler's example by confirming these outstanding cases.

1. INTRODUCTION

In his seminal work [Z89], Ziegler introduced the concept of a multi-arrangement generalizing the notion of a hyperplane arrangement. A natural example of such a multi-arrangement is the reflection arrangement of an irreducible unitary reflection group with multiplicity given by the number of reflections associated with each hyperplane. Ziegler showed in [Z89] that each such multi-reflection arrangement is free with the possible exception of just three instances. In this short note we revisit Ziegler's example and show by computational means that these remaining cases are also free in Theorem 2.1.

Ever since Ziegler's introduction of multi-arrangements, the subject flourished. In particular, the question of freeness of multi-arrangements is a very active field of research, e.g. see the recent work [ATW08] and [Y14] and the references therein.

1.1. Multi-Arrangements. Let \mathbb{K} be a field and let $V = \mathbb{K}^\ell$. Let $\mathcal{A} = (\mathcal{A}, V)$ be a central ℓ -arrangement of hyperplanes in V . A *multi-arrangement* is a pair (\mathcal{A}, ν) consisting of a hyperplane arrangement \mathcal{A} and a *multiplicity* function $\nu : \mathcal{A} \rightarrow \mathbb{Z}_{\geq 0}$ associating to each hyperplane H in \mathcal{A} a non-negative integer $\nu(H)$. Alternately, the multi-arrangement (\mathcal{A}, ν) can also be thought of as the multi-set of hyperplanes

$$(\mathcal{A}, \nu) = \{H^{\nu(H)} \mid H \in \mathcal{A}\}.$$

The order of \mathcal{A} is the cardinality $|\mathcal{A}|$ of the set \mathcal{A} and the *order* of the multi-arrangement (\mathcal{A}, ν) is the cardinality of the multi-set (\mathcal{A}, ν) , we write $|\nu| := |(\mathcal{A}, \nu)| = \sum_{H \in \mathcal{A}} \nu(H)$. For a multi-arrangement (\mathcal{A}, ν) , the underlying arrangement \mathcal{A} is sometimes called the associated *simple* arrangement, and so (\mathcal{A}, ν) itself is simple if and only if $\nu(H) = 1$ for each $H \in \mathcal{A}$.

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1.2. Freeness of Arrangements and Multi-Arrangements. Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V . If x_1, \dots, x_ℓ is a basis of V^* , then we identify S with the polynomial ring $\mathbb{K}[x_1, \dots, x_\ell]$. Letting S_p denote the \mathbb{K} -subspace of S consisting of the homogeneous polynomials of degree p (along with 0), S is naturally \mathbb{Z} -graded: $S = \bigoplus_{p \in \mathbb{Z}} S_p$, where $S_p = 0$ in case $p < 0$.

Let $\text{Der}(S)$ be the S -module of algebraic \mathbb{K} -derivations of S . For $i = 1, \dots, \ell$, let $D_i := \partial/\partial x_i$ be the usual derivation of S . Then D_1, \dots, D_ℓ is a \mathbb{K} -basis of $\text{Der}(S)$. We say that $\theta \in \text{Der}(S)$ is *homogeneous of polynomial degree p* provided $\theta = \sum_{i=1}^{\ell} f_i D_i$, where $f_i \in S_p$ for each $1 \leq i \leq \ell$. In this case we write $\text{pdeg } \theta = p$. Let $\text{Der}(S)_p$ be the \mathbb{K} -subspace of $\text{Der}(S)$ consisting of all homogeneous derivations of polynomial degree p . Then $\text{Der}(S)$ is a graded S -module: $\text{Der}(S) = \bigoplus_{p \in \mathbb{Z}} \text{Der}(S)_p$.

Let \mathcal{A} be an arrangement in V . Then for $H \in \mathcal{A}$ we fix $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. The *defining polynomial* $Q(\mathcal{A})$ of \mathcal{A} is given by $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$.

The *module of \mathcal{A} -derivations* of \mathcal{A} is defined by

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for each } H \in \mathcal{A}\}.$$

We say that \mathcal{A} is *free* if the module of \mathcal{A} -derivations $D(\mathcal{A})$ is a free S -module.

With the \mathbb{Z} -grading of $\text{Der}(S)$, the module of \mathcal{A} -derivations becomes a graded S -module $D(\mathcal{A}) = \bigoplus_{p \in \mathbb{Z}} D(\mathcal{A})_p$, where $D(\mathcal{A})_p = D(\mathcal{A}) \cap \text{Der}(S)_p$, [OT92, Prop. 4.10]. If \mathcal{A} is a free arrangement, then the S -module $D(\mathcal{A})$ admits a basis of ℓ homogeneous derivations, say $\theta_1, \dots, \theta_\ell$, [OT92, Prop. 4.18]. While the θ_i 's are not unique, their polynomial degrees $\text{pdeg } \theta_i$ are unique (up to ordering). This multiset is the set of *exponents* of the free arrangement \mathcal{A} and is denoted by $\exp \mathcal{A}$.

Following Ziegler [Z89], we extend this notion of freeness to multi-arrangements. The *defining polynomial* $Q(\mathcal{A}, \nu)$ of the multi-arrangement (\mathcal{A}, ν) is given by

$$Q(\mathcal{A}, \nu) := \prod_{H \in \mathcal{A}} \alpha_H^{\nu(H)},$$

a polynomial of degree $|\nu|$ in S .

The *module of \mathcal{A} -derivations* of (\mathcal{A}, ν) is defined by

$$D(\mathcal{A}, \nu) := \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^{\nu(H)} S \text{ for each } H \in \mathcal{A}\}.$$

We say that (\mathcal{A}, ν) is *free* if $D(\mathcal{A}, \nu)$ is a free S -module, [Z89, Def. 6].

As in the case of simple arrangements, if (\mathcal{A}, ν) is free, there is a homogeneous basis $\theta_1, \dots, \theta_\ell$ of $D(\mathcal{A}, \nu)$. The multi-set of the unique polynomial degrees $\text{pdeg } \theta_i$ are the *multi-exponents* of the free multi-arrangement (\mathcal{A}, ν) and is denoted by $\exp(\mathcal{A}, \nu)$. It follows from Ziegler's analogue of Saito's criterion [Z89, Thm. 8] that $\sum \text{pdeg } \theta_i = \deg Q(\mathcal{A}, \nu) = |\nu|$.

As is the case for simple arrangements, if ℓ is at most 2, then (\mathcal{A}, ν) is free, [Z89, Cor. 7].

2. ZIEGLER'S MULTI-ARRANGEMENT FOR UNITARY REFLECTION GROUPS

Now let $\mathbb{K} = \mathbb{C}$, the complex numbers. Suppose that W is a finite, unitary reflection group acting on the complex vector space V . Let $\mathcal{A}(W) = (\mathcal{A}(W), V)$ be the associated hyperplane arrangement of W , the *reflection arrangement* of W . For $w \in W$, we write $\text{Fix}(w) := \{v \in V \mid wv = v\}$ for the fixed point subspace of w . We use the classification and labeling of the irreducible unitary reflection groups due to Shephard and Todd, [ST54].

Ziegler defined the multi-arrangement $(\mathcal{A}(W), \varrho)$ of W , with the *reflection multiplicity* ϱ , i.e.

$$\varrho(H) := |\{w \in W \mid \text{Fix}(w) = H\}|$$

is the number of pseudo-reflections having H as fixed point hyperplane. So $|\varrho|$ is the number of reflections in W and the defining polynomial $Q(\mathcal{A}(W), \varrho)$ of $(\mathcal{A}(W), \varrho)$ is the determinant of the Jacobian of a fixed set of basic invariants of W , cf. [OT92, Thm. 6.42].

Our aim is to complete the proof of the following

Theorem 2.1. *For W a finite, unitary reflection group, the multi-arrangement $(\mathcal{A}(W), \varrho)$ of W is free.*

Proof. A product of multi-arrangements is free if and only if each factor is free: using [ATW08, Lem. 1.3], the proof of [OT92, Thm. 4.28] readily extends to multi-arrangements, thanks to Ziegler's analogue of Saito's criterion [Z89, Thm. 8]. Thus we may assume that W is irreducible.

All but three cases were proved by Ziegler in [Z89]. If W is generated by pseudo-reflections of order 2, e.g. if W is a Coxeter group, then $\varrho \equiv 1$. Thus in these instances $(\mathcal{A}(W), \varrho) = \mathcal{A}(W)$ is simple. In these cases $\mathcal{A}(W)$ is known to be free, thanks to Terao's work, [T80]. If W is cyclic or of rank 2, then $(\mathcal{A}(W), \varrho)$ is free, by [Z89, Cor. 7]. Also for the monomial groups $W = G(r, p, \ell)$, Ziegler showed that $(\mathcal{A}(W), \varrho)$ is free, [Z89, Ex. 15].

So the question about freeness of $(\mathcal{A}(W), \varrho)$ is only outstanding for the three exceptional groups $W = G_{25}, G_{26}$ and G_{32} . Both G_{25} and G_{32} are generated by pseudo-reflections of order 3, while G_{26} admits 9 pseudo-reflections of order 2 and 24 of order 3. Therefore, we have $|\mathcal{A}(W)| = 12, 21, 40$ and $|(\mathcal{A}(W), \varrho)| = 24, 33, 80$, respectively.

Our proof of these remaining cases for Theorem 2.1 is computational. First we use the functionality for complex reflection groups provided by the CHEVIE package in GAP (and some GAP code by J. Michel) (see [S⁺97] and [GHL⁺96]) in order to obtain explicit linear functionals α_H so that $H = \ker \alpha_H$ for the underlying reflection arrangement $\mathcal{A}(W)$. These then allow us to implement the S -module $D(\alpha_H, \varrho) := \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H^{\varrho(H)} S\}$ associated with α_H in the SINGULAR computer algebra system (cf. [GPS09]). Then the module theoretic functionality of SINGULAR is used to show that the modules of derivations in question

$$D(\mathcal{A}(W), \varrho) = \bigcap_{H \in \mathcal{A}(W)} D(\alpha_H, \varrho)$$

are free. In particular, for $W = G_{25}, G_{26}$ or G_{32} , the multi-exponents are $\exp(\mathcal{A}(W), \varrho) = \{8, 8, 8\}, \{10, 10, 13\}$ and $\{20, 20, 20, 20\}$, respectively. As an illustration, we give explicit S -bases for $D(\mathcal{A}(G_{25}), \varrho)$ and $D(\mathcal{A}(G_{26}), \varrho)$ in the next section. Not unexpectedly, they are not particularly enlightening. \square

Note that even though the simple arrangement $\mathcal{A}(W)$ is free [T80], for an arbitrary multiplicity ν of $\mathcal{A}(W)$, the multi-arrangement $(\mathcal{A}(W), \nu)$ need not be free in general, cf. [ATW08, Ex. 5.13].

While our calculations combined with the existing known instances determined by Ziegler do provide a proof of Theorem 2.1, it would nevertheless be very desirable to have a uniform, conceptual proof free of case-by-case considerations and free of computer calculations.

3. DEFINING POLYNOMIALS AND BASES OF $D(\mathcal{A}(W), \varrho)$

To illustrate our computations, we list explicit S -bases for $D(\mathcal{A}(G_{25}), \varrho)$ and $D(\mathcal{A}(G_{26}), \varrho)$. Let x, y , and z be the indeterminates of S and $D_x = \partial/\partial x$, $D_y = \partial/\partial y$, and $D_z = \partial/\partial z$ and let ζ be a primitive 3rd root of unity.

$$\begin{aligned} Q(\mathcal{A}(G_{25}), \varrho) &= Q(G_{25})^3 = (xyz(x+y+z)(x+y+\zeta z)(x+y-(\zeta+1)z) \\ &\quad (x+\zeta y+z)(x+\zeta y+\zeta z)(x+\zeta y-(\zeta+1)z) \\ &\quad (x-(\zeta+1)y+z)(x-(\zeta+1)y+\zeta z)(x-(\zeta+1)y-(\zeta+1)z))^3. \end{aligned}$$

$$\begin{aligned} D(\mathcal{A}(G_{25}), \varrho) &= S((6x^7z + 42x^4y^3z - 21x^4z^4)D_x + (42x^3y^4z + 6y^7z - 21y^4z^4)D_y \\ &\quad + (14x^6z^2 + 28x^3y^3z^2 + 14y^6z^2 - 14x^3z^5 - 14y^3z^5 - z^8)D_z) \\ &\quad + S((6x^7y - 21x^4y^4 + 42x^4yz^3)D_x + (42x^3yz^4 - 21y^4z^4 + 6yz^7)D_z \\ &\quad + (14x^6y^2 - 14x^3y^5 - y^8 + 28x^3y^2z^3 - 14y^5z^3 + 14y^2z^6)D_y) \\ &\quad + S((x^8 + 14x^5y^3 - 14x^2y^6 + 14x^5z^3 - 28x^2y^3z^3 - 14x^2z^6)D_x \\ &\quad + (21x^4y^4 - 6xy^7 - 42xy^4z^3)D_y + (21x^4z^4 - 42xy^3z^4 - 6xz^7)D_z). \end{aligned}$$

$$\begin{aligned} Q(\mathcal{A}(G_{26}), \varrho) &= (y-z)^2(x-z)^2(x-y)^2(y-\zeta z)^2(x-\zeta z)^2(x-\zeta y)^2(y+(\zeta+1)z)^2 \\ &\quad (x+(\zeta+1)y)^2(x+(\zeta+1)z)^2x^3y^3z^3(x+y+z)^3(x+(-\zeta-1)y+\zeta z)^3 \\ &\quad (x+y+\zeta z)^3(x+y+(-\zeta-1)z)^3(x+\zeta y+z)^3(x+(-\zeta-1)y+z)^3 \\ &\quad (x+\zeta y+(-\zeta-1)z)^3(x+(-\zeta-1)y+(-\zeta-1)z)^3(x+\zeta y+\zeta z)^3. \end{aligned}$$

$$\begin{aligned}
D(\mathcal{A}(G_{26}), \varrho) = & S((11x^8yz + 7x^5y^4z + 14x^2y^7z + 7x^5yz^4 + 28x^2y^4z^4 + 14x^2yz^7)D_x \\
& + (14x^7y^2z + 7x^4y^5z + 11xy^8z + 28x^4y^2z^4 + 7xy^5z^4 + 14xy^2z^7)D_y \\
& + (14x^7yz^2 + 28x^4y^4z^2 + 14xy^7z^2 + 7x^4yz^5 + 7xy^4z^5 + 11xyz^8)D_z) \\
& + S((x^{10} + 8x^7y^3 + 7x^4y^6 + 8x^7z^3 - 112x^4y^3z^3 + 7x^4z^6)D_x \\
& + (7x^6y^4 + 8x^3y^7 + y^{10} - 112x^3y^4z^3 + 8y^7z^3 + 7y^4z^6)D_y \\
& + (7x^6z^4 - 112x^3y^3z^4 + 7y^6z^4 + 8x^3z^7 + 8y^3z^7 + z^{10})D_z) \\
& + S((-75x^7y^6 - 21x^4y^9 - 12x^7y^3z^3 + 588x^4y^6z^3 - 75x^7z^6 + 588x^4y^3z^6 - 21x^4z^9)D_x \\
& + (14x^9y^4 - 70x^6y^7 - 35x^3y^{10} - 5y^{13} + 28x^6y^4z^3 \\
& + 588x^3y^7z^3 - 40y^{10}z^3 + 623x^3y^4z^6 - 110y^7z^6 - 21y^4z^9)D_y \\
& + (14x^9z^4 + 28x^6y^3z^4 + 623x^3y^6z^4 - 21y^9z^4 - 70x^6z^7 \\
& + 588x^3y^3z^7 - 110y^6z^7 - 35x^3z^{10} - 40y^3z^{10} - 5z^{13})D_z).
\end{aligned}$$

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